The Cornell Note-taking System

2 1/2"	6"
Cue Column	**Notetaking Column** 1. **Record:** During the lecture, use the notetaking column to record the lecture using telegraphic sentences. 2. **Questions:** As soon after class as possible, formulate questions based on the notes in the right-hand column. Writing questions helps to clarify meanings, reveal relationships, establish continuity, and strengthen memory. Also, the writing of questions sets up a perfect stage for exam-studying later. 3. **Recite:** Cover the notetaking column with a sheet of paper. Then, looking at the questions or cue-words in the question and cue column only, say aloud, in your own words, the answers to the questions, facts, or ideas indicated by the cue-words. 4. **Reflect:** Reflect on the material by asking yourself questions, for example: "What's the significance of these facts? What principle are they based on? How can I apply them? How do they fit in with what I already know? What's beyond them? 5. **Review:** Spend at least ten minutes every week reviewing all your previous notes. If you do, you'll retain a great deal for current use, as well as, for the exam.

2"

Summary

After class, use this space at the bottom of each page to summarize the notes on that page.

Adapted from How to Study in College 7/e by Walter Pauk, 2001 Houghton Mifflin Company

If we let h tend to zero (from one side or the other), then ideally the point $(x+h, f(x+h))$ slides along the curve toward $(x, f(x))$, $x+h$ tends to x, $f(x+h)$ tends to $f(x)$, and the slope of the secant

$$(*) \qquad \frac{f(x+h)-f(x)}{h}$$

tends to a limit that we denote by $f'(x)$†. While $(*)$ represents the slope of the approaching secant, the number $f'(x)$ represents the slope of the graph at the point $(x, f(x))$.

What we call "differential calculus" is the implementation of this idea.

Derivatives and Differentiation

DEFINITION 3.1.1

A function f is said to be differentiable at x if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{exists.}$$

If this limit exists, it is called the *derivative of f at x* and is denoted by $f'(x)$.

As indicated in the introduction, the derivative

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

represents *slope of the graph of f at the point* $(x, f(x))$. The line that passes through the point $(x, f(x))$ with slope $f'(x)$ is called the *tangent line* at the point $(x, f(x))$. (This line is marked by dashes in Figure 3.1.1.)

Example 1 We begin with a linear function

$$f(x)=mx+b\,.$$

The graph of this function is the line $y=mx+b$, a line with constant slope m. We therefore expect $f'(x)$ to be constantly m. Indeed it is: for $h \neq 0$,

$$\frac{f(x+h)-f(x)}{h}=\frac{[m(x+h)+b]-[mx+b]}{h}=\frac{mh}{h}=m$$

and therefore

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0} m = m. \quad \square$$

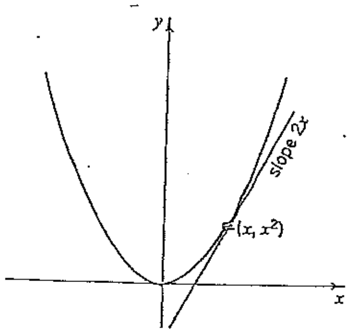


Figure 3.1.2

Example 2 Now we look at the squaring function

$$f(x)=x^2. \qquad \text{(Figure 3.1.2)}$$

To find $f'(x)$, we form the difference quotient

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-x^2}{h}$$

†This prime notation goes back to the French mathematician Joseph-Louis Lagrange (1736–1813). Other notations are introduced later.

If we let h tend to zero (from one side or the other), then ideally the point $(x+h,\ f(x+h))$ slides along the curve toward $(x,\ f(x))$, $x+h$ tends to x, $f(x+h)$ tends to $f(x)$, and the slope of the secant

slope of the secant f $(x, f(x))$ and $(x+h, f(x+h))$

$$(*)\qquad \frac{f(x+h)-f(x)}{h}$$

tends to a limit that we denote by $f'(x)$†. While $(*)$ represents the slope of the approaching secant, the number $f'(x)$ represents the slope of the graph at the point $(x,\ f(x))$.

What we call "differential calculus" is the implementation of this idea.

Derivatives and Differentiation

definition of a derivative, $f'(x)$

DEFINITION 3.1.1

A function f is said to be differentiable at x if

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{exists.}$$

If this limit exists, it is called the *derivative of f at x* and is denoted by $f'(x)$.

• Slope of graph of f at the point $(x, f(x))$

As indicated in the introduction, the derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

represents *slope of the graph of f at the point* $(x,\ f(x))$. The line that passes through the point $(x,\ f(x))$ with slope $f'(x)$ is called the *tangent line* at the point $(x,\ f(x))$. (This line is marked by dashes in Figure 3.1.1.)

• Example:
) pt and close pt
, $f'(x)$) $(x+h, f(x+h))$
) slope btwn 2 pts
) $\lim_{h\to 0}$ slope
) $f'(x) = m$

Example 1 We begin with a linear function

$$f(x) = mx + b\,.$$

The graph of this function is the line $y = mx + b$, a line with constant slope m. We therefore expect $f'(x)$ to be constantly m. Indeed it is: for $h \neq 0$,

$$\frac{f(x+h)-f(x)}{h} = \frac{[m(x+h)+b]-[mx+b]}{h} = \frac{mh}{h} = m$$

and therefore

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} m = m. \quad \square$$

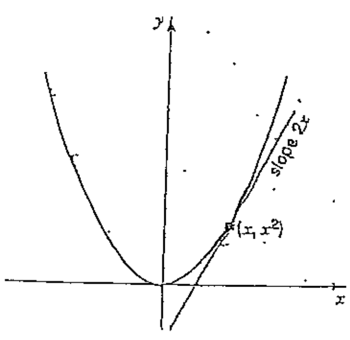


Figure 3.1.2

Example 2 Now we look at the squaring function

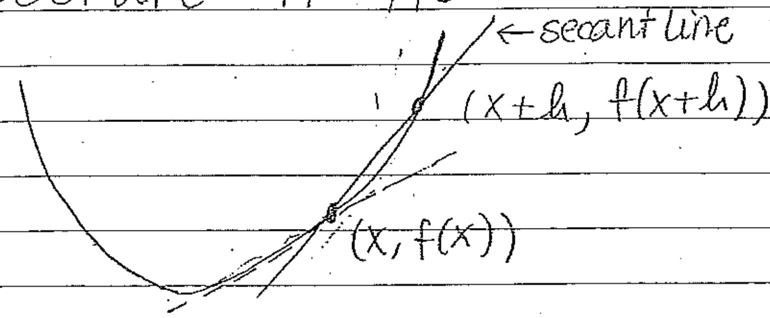
$$f(x) = x^2. \qquad \text{(Figure 3.1.2)}$$

To find $f'(x)$, we form the difference quotient

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h}$$

† This prime notation goes back to the French mathematician Joseph-Louis Lagrange (1736–1813). Other notations are introduced later.

Lecture 9/10/10



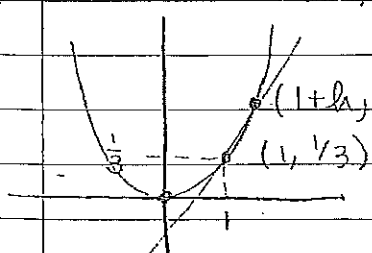
As $h \to 0$, secant line → tangent line
slope of secant lines → [slope of the graph at x]

$$\boxed{\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)}$$

When this limit exists at $x = c$, we say $f(x)$ is differentiable at c.

Example

$f(x) = \frac{1}{3}x^2$, find $f'(1)$, find the tangent line at $(1, f(1)) = (1, 1/3)$, find $f'(x)$



Slope of the secant: $\dfrac{f(1+h) - f(1)}{h}$

$$= \frac{\frac{1}{3}(1+h)^2 - \frac{1}{3}}{h} = \frac{\frac{1}{3}(1+2h+h^2) - \frac{1}{3}}{h}$$

① $f'(1) = \lim_{h \to 0} \frac{2}{3} + \frac{1}{3}h$ ⇐

$$= \frac{2}{3} + \frac{1}{3}h \quad \leftarrow \text{slope of the secant line}$$

$\boxed{f'(1) = \frac{2}{3}}$ slope of the tangent line at $x=1$

② Find the tangent line at $x = 1$

pt $(1, 1/3)$ Slope $= 2/3$ or $y - \frac{1}{3} = \frac{2}{3}(x-1)$

$1/3 = 2/3 + b$ $b = -1/3$ ⇒ tangent line: $y = 2/3 x - 1/3$

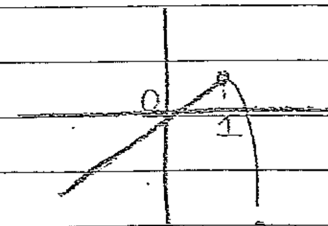
③ Find $f'(x)$

$$\lim_{h \to 0} \frac{\frac{1}{3}(x+h)^2 - \frac{1}{3}x^2}{h} = \lim_{h \to 0} \frac{\frac{1}{3}x^2 + \frac{2}{3}xh + h^2 - \frac{1}{3}x^2}{h} = \lim_{h \to 0} \frac{2}{3}x + h$$

$\boxed{2x}$

Lecture 9/10/10

Example
$$f(x) = \begin{cases} 4 - x^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$



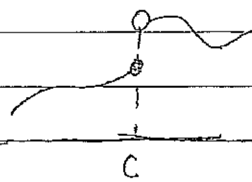
f(x)

$$f'(x) = \begin{cases} -2x & x > 1 \\ 3 & x < 1 \end{cases}$$

$f'(1)$ from left $= 3$, from right $= -2$

NON differentiable. LHL≠RHL

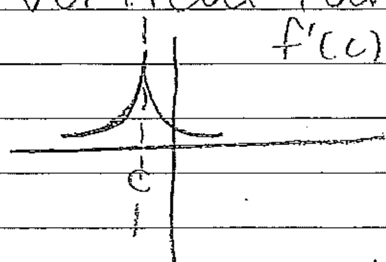
Discontinuity ⇒ NON differentiable.

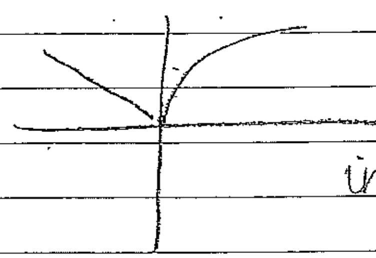


$f'(c)$ does not exist

Vertical Tangent or infinite slope ⇒ Nondiff.

$f'(c)$ DNE

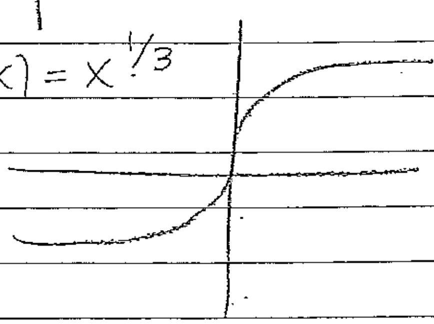




$$f(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -x & x < 0 \end{cases}$$

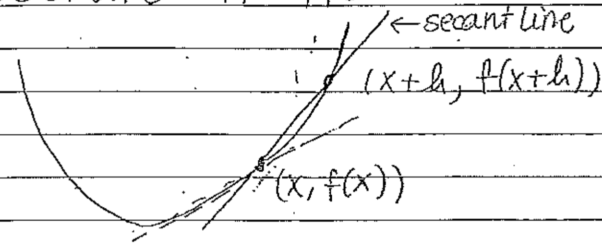
infinite slope
⇒ nondiff.

$f(x) = x^{1/3}$



$f'(0)$ DNE

Lecture 9/10/10



As $h \to 0$, secant line → tangent line

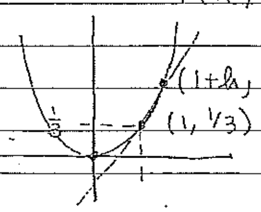
slope of secant lines → [slope of the graph at x]

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)}$$

When this limit exists at $x = c$, we say $f(x)$ is differentiable at c.

Example

$f(x) = \frac{1}{3}x^2$, find $f'(1)$, find the tangent line at $(1, f(1)) = (1, 1/3)$, find $f'(x)$



Slope of the secant: $\dfrac{f(1+h) - f(1)}{h}$

$$= \frac{\frac{1}{3}(1+h)^2 - \frac{1}{3}}{h} = \frac{\frac{1}{3}(1 + 2h + h^2) - \frac{1}{3}}{h}$$

$$= \frac{2}{3} + \frac{1}{3}h \quad \leftarrow \text{slope of the secant line}$$

① $f'(1) = \lim_{h \to 0} \frac{2}{3} + \frac{1}{3}h \quad \Leftarrow$

$\boxed{f'(1) = \frac{2}{3}}$ slope of the tangent line at $x=1$

② Find the tangent line at $x=1$

pt $(1, 1/3)$ Slope $= 2/3$ or $y - \frac{1}{3} = \frac{2}{3}(x-1)$

$1/3 = 2/3 + b$ $b = -1/3 \Rightarrow$ tangent line: $y = 2/3\,x - 1/3$

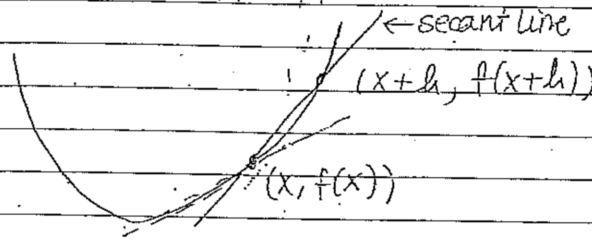
③ Find $f'(x)$

$$\lim_{h \to 0} \frac{\frac{1}{3}(x+h)^2 - \frac{1}{3}x^2}{h} = \lim_{h \to 0} \frac{\frac{1}{3}x^2 + \frac{2}{3}xh + h^2 - \frac{1}{3}x^2}{h} = \lim_{h \to 0} \frac{2}{3}x + h$$

$$= \boxed{\frac{2}{3}x}$$

Lecture 9/10/10

slope of secant



As $h \to 0$, secant line → tangent line
slope of secant lines ↳ [slope of the graph at x]

definition of derivative

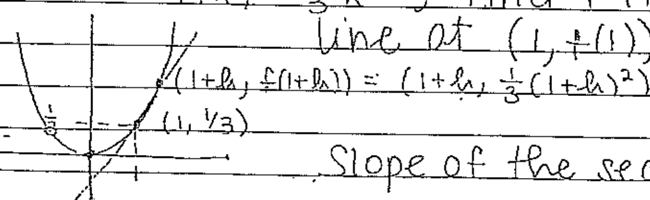
$$\boxed{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)}$$

When this limit exists at $x=c$, we say $f(x)$ is differentiable at c.

Example:
Finding the tangent line at a point.

Example

$f(x) = \frac{1}{3}x^2$, find $f'(1)$, find the tangent line at $(1, f(1)) = (1, 1/3)$, find $f'(x)$



Slope of the secant: $\dfrac{f(1+h)-f(1)}{h}$

$$= \frac{\frac{1}{3}(1+h)^2 - \frac{1}{3}}{h} = \frac{\frac{1}{3}(1+2h+h^2) - \frac{1}{3}}{h}$$

① $f'(1) = \lim_{h\to 0} \frac{2}{3} + \frac{1}{3}h \Leftarrow$

$= \frac{2}{3} + \frac{1}{3}h$ ← slope of the secant line

$\boxed{f'(1) = \frac{2}{3}}$ slope of the tangent line at $x=1$

② Find the tangent line at $x=1$

pt $(1, 1/3)$ slope $= 2/3$ or $y - \frac{1}{3} = \frac{2}{3}(x-1)$

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③ Find $f'(x)$

$$\lim_{h\to 0} \frac{\frac{1}{3}(x+h)^2 - \frac{1}{3}x^2}{h} = \lim_{h\to 0} \frac{\frac{1}{3}x^2 + \frac{2}{3}xh + h^2 - \frac{1}{3}x^2}{h} = \lim_{h\to 0} \frac{2}{3}x + h = \boxed{\frac{2}{3}x}$$

As the distance h between x and $x+h$ goes to zero, the secant line becomes the tangent line.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

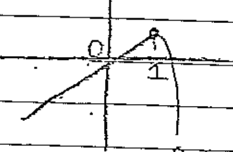
To find $f'(1)$ for $f(x) = \frac{1}{3}x^2$

① Find tangent line eqn. pt $(1, f(1))$

• Example: RHL $\stackrel{?}{=}$ LHL

Lecture 9/10/10

Example $f(x) = \begin{cases} 4 - x^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$



$f(x)$

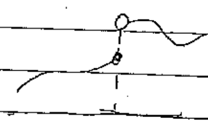
$f'(x) = \begin{cases} -2x & x > 1 \\ 3 & x < 1 \end{cases}$

$f'(1)$ from left $= 3$, from right $= -2$

NON differentiable. LHL $\neq$ RHL

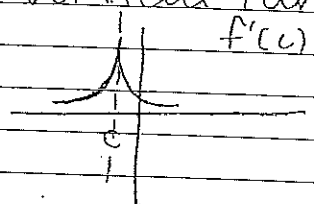
• Nondifferentiable Examples

Discontinuity $\Rightarrow$ NON differentiable.

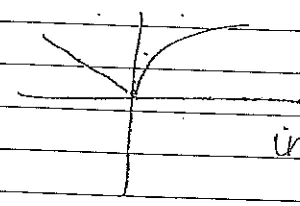


$f'(c)$ does not exist

Vertical Tangent or infinite slope $\Rightarrow$ Nondiff.



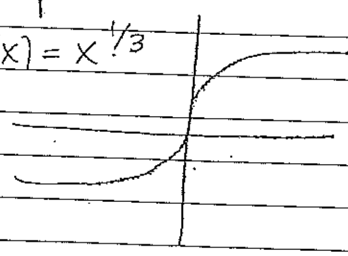
$f'(c)$ DNE



$f(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -x & x < 0 \end{cases}$

infinite slope $\Rightarrow$ nondiff.

$f(x) = x^{1/3}$



$f'(0)$ DNE

* If the RHL is different than the LHL, you cannot find the derivative at that point $\Rightarrow$ slopes are different $\Rightarrow$ no derivative

* Examples of nondifferentiable points:
  1. infinite discontinuity
  2. vertical tangent
  3. vertical cusp (you cannot take the derivative)